On the stability of the replica symmetric theory of the matching problem: the longitudinal sector

1987 J. Phys. A: Math. Gen. 20 L841

(http://iopscience.iop.org/0305-4470/20/13/006)

J. Phys. A: Math. Gen. 20 (1987) L841-L845. Printed in the UK

LETTER TO THE EDITOR

On the stability of the replica symmetric theory of the matching problem: the longitudinal sector

C De Dominicis† and P Mottishaw‡

† Service de Physique Théorique, CEN Saclay, 91191 Gif-sur-Yvette, Cedex, France
‡ Department of Theoretical Physics, University of Oxford, 1 Keble Road, Oxford OX1 3NP, UK

Received 5 May 1987

Abstract. A replica symmetric theory of the matching problem was given by Orland and by Mézard and Parisi. Here we investigate the stability of this solution. In the zero-temperature limit we find that the longitudinal spectrum is continuous and lies in the interval $0<\lambda<2$. We conclude that the solution of Mézard and Parisi is stable to longitudinal fluctuations. We give the explicit form of the longitudinal eigenvectors. The method used lends itself to dealing with replica symmetry broken solutions where necessary.

In recent years there has been a great deal of interest in the application of statistical mechanics to optimisation problems (see, for example, [1]). The replica method [2] has been applied to a number of important optimisation problems [3-5]. Except in particularly simple cases [6], work has been restricted to the replica symmetric ansatz. This is rather unfortunate since it has often been suggested that for hard (NP-complete) optimisation problems replica symmetry must be broken. In this letter we consider the matching problem [7]. That is: given $2N$ points, how can one pair them in order to minimise the sum of the distances between each pair? This problem is polynomially bounded [7]. The replica method has been applied to this problem by Orland [4] and Mézard and Parisi [3]. Here we investigate the stability of the replica symmetric ansatz used by these authors.

As our starting point we shall use the expression for $\overline{Z^n}$ derived by Orland [4] and Mézard and Parisi [3]:

$$\overline{Z^n}=\int\prod_{p=1}\prod_{a_1<\ldots<a_p}\left(\frac{\mathrm{d}Q_{a_1\ldots a_p}}{\sqrt{2\pi N g_p}}\right)\exp(N\{-\hat{\beta}\tilde{f}_n\}) \tag{1}$$

where

$$-\hat{\beta}\tilde{f}_n=-\frac{1}{2}\sum_{p=1}^{n}\frac{1}{g_p}\sum_{a_1<\ldots<a_p}(Q_{a_1\ldots a_p})^2+2\log z \tag{2}$$

$$z=\int_0^{2\pi}\frac{\mathrm{d}\lambda^1}{2\pi}\cdots\int_0^{2\pi}\frac{\mathrm{d}\lambda^n}{2\pi}\exp\left(\mathrm{i}\sum_{a=1}^{n}\lambda^a+\sum_{p=1}^{n}\sum_{a_1<\ldots<a_p}Q_{a_1\ldots a_p}\exp[-\mathrm{i}(\lambda_{a_1}+\ldots+\lambda_{a_p})]\right) \tag{3}$$

and

$$g_p=(p\hat{\beta})^{-(r+1)}. \tag{4}$$

Here the $Q_{a_1...a_p}$ are $(2^n - 1)$-order parameters generalising the Edwards-Anderson [2] parameters $Q_{a_1a_2}$. Following [8] we introduce a new set of order parameters defined by

$$X(\{\sigma_a\}) = \sum_{p=1}^{n} \sum_{a_1<...<a_p} Q_{a_1...a}\sigma_{a_1} \ldots \sigma_{a_p} \tag{5}$$

where $\sigma_a = \pm 1$, $a = 1, 2, \ldots, n$. There is a distinct $X(\{\sigma_a\})$ for each configuration $\{\sigma_a\}$ of Ising spins. This gives 2^n parameters and therefore we include the constraint

$$\operatorname*{Tr}_{\sigma_a} X(\{\sigma_a\}) = 0 \tag{6}$$

to reduce this to $(2^n - 1)$ independent parameters. This generalised order parameter (with $2^n - 1$ components) is particularly suited to represent any system described by the $q_{a_1...a_p}$, with or without replica symmetry breaking.

In terms of the X parameters, equations (2) and (3) become

$$-\hat{\beta}\tilde{f}_n = -2^{-(2n+1)} \operatorname*{Tr}_{\tau_a} X(\tau_a) \operatorname*{Tr}_{\sigma_a} X(\sigma_a) K\left(\sum_a \tau_a\sigma_a\right) + \gamma\left(\operatorname*{Tr}_{\sigma_a} X(\sigma_a)\right)^2 - 2^{1-n} \operatorname*{Tr}_{\sigma_a} X(\sigma_a) + 2\log z \tag{7}$$

where

$$z = \sum_{t=0}^{\infty} \frac{2^{-nt}}{t!} \operatorname*{Tr}_{\sigma_a^1} \ldots \operatorname*{Tr}_{\sigma_a^t} X(\sigma_a^1) \ldots X(\sigma_a^t) \prod_{a=1}^{n} \left(\sum_{s=1}^{t} \sigma_a^s\right) \tag{8}$$

and

$$K\left(\sum_a^n \nu_a\right) = \sum_{p=1}^{n} (p\hat{\beta})^{r+1} \sum_{a_1<...<a_p} \nu_{a_1} \ldots \nu_{a_p}. \tag{9}$$

The constraint (6) is imposed by taking $\gamma \to -\infty$.

The first derivative of (7) with respect to $X(\{\nu_a\})$ gives the equation of motion

$$0 = -2^{-n} \operatorname*{Tr}_{\tau_a} X(\tau_a) K\left(\sum \tau_a \nu_a\right) + \frac{2}{z} \sum_{t=0}^{\infty} \frac{2^{-nt}}{t!} \operatorname*{Tr}_{\sigma_a^1} \ldots \operatorname*{Tr}_{\sigma_a^t} X(\sigma_a^1) \ldots X(\sigma_a^t) \times \left[\prod_{a=1}^{n} \left(\nu_a + \sum_{s=1}^{t} \sigma_a^s\right) - \prod_{a=1}^{n} \left(\sum_{s=1}^{t} \sigma_a^s\right)\right] \tag{10}$$

where we have made use of the constraint (6).

We now make the replica symmetric ansatz that $X(\{\sigma_a\})$ depends only on the total spin

$$\hat{\sigma} = \sum_{a=1}^{n} \sigma_a \tag{11}$$

of the spin configuration $\{\sigma_a\}$. In our previous work [8, 9] we have shown that in the $n \to 0$ limit $\hat{\sigma}$ can be analytically continued to the pure imaginary axis. Using the methods discussed there, we have shown that the function $X(\hat{\sigma})$ is related to the function $\hat{G}(x)$ of Mézard and Parisi [3] by

$$\hat{G}(x) = \int_{-i\infty}^{i\infty} ds\, X(s) \int_{-\infty}^{\infty} \frac{du}{2\pi i} \exp(-us)[1 - \exp(-e^{\hat{\beta}x} + \tanh u)]. \tag{12}$$

Likewise, an inverse formula expressing $X(s)$ in terms of $\hat{G}(x)$ may also be derived. Substituting (12) in (10) we find in the $n \to 0$ limit the replica symmetric equation of [3, 4]:

$$\hat{G}(x) = \frac{2}{\hat{\beta}^r} \int_{-\infty}^{\infty} \mathrm{d}y \exp(-\hat{G}(y)) B_r(\hat{\beta}(x+y)) \tag{13a}$$

where

$$B_r(x) = \sum_{p=1}^{\infty} \frac{(-1)^{p-1} \, \mathrm{e}^{px}}{p^r (p!)^2}. \tag{13b}$$

In the limit $\beta \to \infty$ and with $r = 0$, equation (13a) has the solution [3]

$$\hat{G}(x) = \ln(1 + \mathrm{e}^{2x}). \tag{13c}$$

In the following we consider the stability of this solution to both longitudinal and transverse fluctuations.

Taking the derivative of equation (7) with respect to $X(\{\sigma_a\})$ and $X(\{\tau_a\})$ we obtain the stability matrix

$$\begin{aligned} M(\{\sigma_a\}, \{\tau_a\}) = {} & -4^{-n} K\left(\sum_a \sigma_a \tau_a\right) + 2\gamma \\ & + z^{-1} 2^{(1-2n)} \sum_{t=0}^{\infty} (1/t! 2^{nt}) \operatorname*{Tr}_{\sigma_a^1} \ldots \operatorname*{Tr}_{\sigma_a^t} X(\sigma_a^1) \ldots X(\sigma_a^t) \\ & \times \left[\prod_{a=1}^{n} \left(\sigma_a + \tau_a + \sum_{s=1}^{t} \sigma_a^s \right) - \prod_{a=1}^{n} \left(\tau_a + \sum_{s=1}^{t} \sigma_a^s \right) \right] \\ & - (1/z 2^{3n}) \left(\operatorname*{Tr}_{\nu_a} X(\nu_a) K\left(\sum_a \sigma_a \nu_a \right) \right) \\ & \times \left(\sum_{t=0}^{\infty} (1/t! 2^{nt}) \operatorname*{Tr}_{\sigma_a^1} \ldots \operatorname*{Tr}_{\sigma_a^t} X(\sigma_a^1) \ldots X(\sigma_a^t) \prod_{a=1}^{n} \left(\tau_a + \sum_{s=1}^{t} \sigma_a^s \right) \right). \end{aligned} \tag{14}$$

The eigenvalue equation is

$$\operatorname*{Tr}_{\tau_a} v(\{\tau_a\}) M(\{\sigma_a\}, \{\tau_a\}) = -\lambda v(\{\sigma_a\}) \tag{15}$$

where the stability condition is that all eigenvalues should be positive.

There always exists the trivial eigenvector where $v(\{\sigma_a\})$ is independent of $\{\sigma_a\}$. The eigenvalue corresponding to this constant eigenvector is -2γ and so the system is always stable to this type of fluctuation. The condition that all remaining eigenvectors are orthogonal to this is

$$\operatorname*{Tr}_{\sigma_a} v(\{\sigma_a\}) = 0. \tag{16}$$

As discussed in [8], when $X(\{\sigma_a\})$ is replica symmetric, the space of eigenvectors is completely spanned by vectors of the form

$$v(\{\sigma_a\}) = v\left(\hat{\sigma}, \sum_a \sigma_a \mu_a\right). \tag{17}$$

Here $\{\mu_a\}$ is a configuration of Ising spins that labels the vector. Vectors with the same $\hat{\mu}$ form degenerate subspaces. To span the space it is sufficient to consider vectors with $\hat{\mu} = n$, $n-2$, $n-4, \ldots, 0$. In this letter we consider only the $\hat{\mu} = n$ subspace. Vectors with $\hat{\mu} = n$ are functions only of $\hat{\sigma}$, and so correspond to longitudinal fluctuations. The study of transverse components is deferred to a later publication.

In the $n \to 0$ limit the longitudinal eigenvector equation becomes

$$\hat{\beta}^r v(x) + 2\int_{-\infty}^{\infty} \mathrm{d}y \exp(-\hat{G}(y)) v(y) B_r(\hat{\beta}(x+y)) = \lambda \int_{-\infty}^{x} \mathrm{d}y_{r+1} \ldots \int_{-\infty}^{y_2} \mathrm{d}y_1\, v(y_1) \tag{18}$$

where the vectors have been transformed in a similar way to $X(\hat{\sigma})$ in equation (12). Note that in the special case of the longitudinal fluctuations, the eigenvalue equation (18) can also be obtained by considering fluctuations in $Q_{a_1 \ldots a_p}$ of the form δQ_p. The relation between δQ_p and $v(x)$ is then

$$v(x) = \sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{p!} \delta Q_p \exp(\hat{\beta} x p). \tag{19}$$

The real importance of the method given here is that it can also deal with transverse fluctuations.

For $r = 0$, and in the limit $\hat{\beta} \to \infty$, equation (18) becomes

$$v(x) + \int_{-x}^{\infty} \mathrm{d}y (1 - \tanh y) v(y) = \lambda \int_{-\infty}^{x} \mathrm{d}y\, v(y) \tag{20}$$

where we have used the solution, equation (13*c*), of [3]. The most general solution of (20) is

$$v(x) = \frac{C}{\cosh x} \{(\lambda - 1) \exp[(\lambda + 1)x] + \lambda \exp[(\lambda - 1)x] + \exp[(1 - \lambda)x]\} \tag{21}$$

with the boundary condition

$$v(-\infty) = 0 \tag{22}$$

and where C is an arbitrary constant. The boundary condition (22) is equivalent to the condition (16) of orthogonality to the constant eigenvector. This condition is satisfied by a continuous spectrum of eigenvalues in the interval

$$0 < \lambda < 2. \tag{23}$$

Therefore the longitudinal part of the spectrum is positive definite, and so the solution (14) of Mézard and Parisi [3] is stable to longitudinal fluctuations.

The eigenvalue equation for transverse fluctuations may be obtained from equations (15) and (17), using similar methods. The transverse vectors are functions of two variables, and so the resulting equation is much harder to solve. We leave this problem to a future publication.

In conclusion we have shown that the replica symmetric solution of the matching problem given by Mézard and Parisi [3] is stable to longitudinal fluctuations.

PM would like to thank R Heimann and R Dewar for useful discussions.

Note added. After the completion of this work we received a preprint from Mézard and Parisi [10] in which they discuss the same problem. They agree with our conclusion that the solution of [3] is stable to longitudinal fluctuations. They also give a complete and detailed analysis of the transverse fluctuations. Our results for the longitudinal case have the advantage of being explicit analytic forms at $T = 0$, rather than numerical solutions at low temperature.

References

[1] Mézard M 1987 *Proc. 1986 Heidelberg Colloq. on Glassy Dynamics and Optimization* (Berlin: Springer)
[2] Edwards S F and Anderson P W 1975 *J. Phys. F: Met. Phys.* **5** 965
[3] Mézard M and Parisi G 1985 *J. Physique Lett.* **46** L771
[4] Orland H 1985 *J. Physique Lett.* **46** L763
[5] Mézard M and Parisi G 1986 *J. Physique* **47** 1285
[6] Fu Y and Anderson P W 1986 *J. Phys. A: Math. Gen.* **19** 1605
[7] Papdimitriou C H and Steiglitz K 1982 *Combinatorial Optimization* (New York: Prentice Hall)
[8] Mottishaw P and De Dominicis C 1987 *J. Phys. A: Math. Gen.* **20** L375
[9] De Dominicis C and Mottishaw P 1987 *Europhys. Lett.* **3** 87; 1987 *Sietges Conf. Proc.* (*Lecture Notes in Physics* **268**) ed L Garrido (Berlin: Springer) p 123
[10] Mézard M and Parisi G 1987 *Preprint* LPTENS